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ON THE CYCLIC BEHAVIOR OF RANDOM TRANSFORMATIONS ON A FINITE SE--ETC(U)

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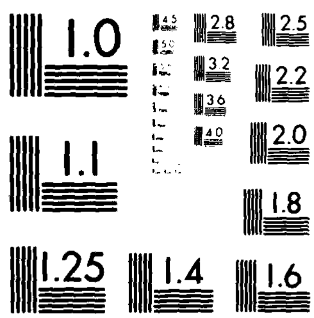

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ON THE CYCLIC BEHAVIOR OF RANDOM TRANSFORMATIONS ON A FINITE SET

By

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ON THE CYCLIC BEHAVIOR OF RANDOM TRANSFORMATIONS ON A FINITE SET

by

Alan E. Gelfand

1. Introduction

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of n elements and let $\mathcal{J}$ be the set of all transformations from X into itself. For $T \in \mathcal{J}$ we take T^k to have its usual meaning. Suppose for any $x \in X$ we look at the sequence $T^j x$, $j = 0,1,2,\ldots$ $(T^0 x \equiv x)$. Since X is finite, given an arbitrary initial element, the sequence T^j must eventually encounter an element it had shown before. Doing so, it must thereafter repeat the intermediate sequence of elements. Such a sequence of elements is called a cycle. The number of distinct elements in the cycle is called the cycle length. For a given x and a given T there will thus be one and only one cycle, say of length r (which we may call the cycle associated with x). Then for any x' on this cycle

$$T^{mr} x' = x' \ , \ m = 0,1,2,\ldots \ .$$

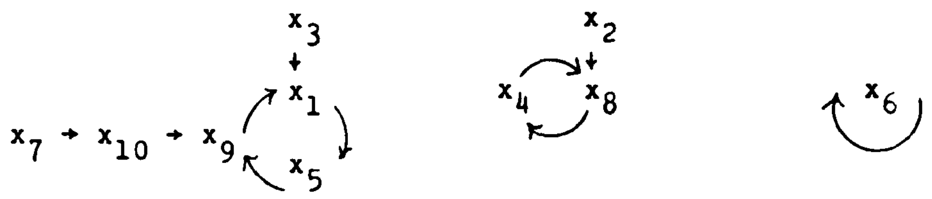
But for a given T not all elements in X must be on a cycle. Some elements may be transient in that they occur during a run-in period prior to T falling into a cycle. Moreover, starting from differing x's may lead T to fall into differing cycles, i.e. there may be many cycles associated with T. This leads to the notion of a cycle space for T. The number of cycles is obviously

between 1 and n as is the number of cyclic elements (i.e., elements on some cycle).

For the transformation T with n = 10 given by

x	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
Tx	x_5	x_8	x_1	x_8	x_9	x_6	x_{10}	x_4	x_1	x_9

we may graphically describe the cycle space as



It is the purpose of this paper to develop a collection of results which effectively describe the cycle space of a randomly selected T . The application of these results to the study of systems having a finite number of states is apparent and for this reason we will use the term "state" interchangeably with the term "element."

The extant literature in this area is quite limited. Gontcharoff in some early work considers the distribution of cycles in permutations of a finite number of elements. Rubin and Sitgreaves, in a very long and detailed article, consider some aspects of the cycle space without formally recognizing it. Harris extends their work and includes some results discussed here but obtained from a different point of view. Katz and his student, Folkert, examine the

expected number of cycles. Cull studies the problem in a system setting (in particular, using binary switching nets although to no particular advantage) and develops some results (with a few errors) on the expected number of cycles and cyclic states.

Our format, then, is as follows. In section 2 we formalize the problem developing convenient notation and definitions. In section 3 we introduce random transformations. In section 4 we demonstrate the advantage of viewing the problem in terms of square arrays of row-exchangeable variables. In section 5 we offer exact results for fixed n and in section 6 we present some attractive asymptotic results.

2. The Setup

Consider again a finite set X with elements $x_1, x_2, \ldots, x_n$. Any transformation $T \in \mathcal{J}$ from X into itself may be given a matrix representation through an $n \times n$ transition matrix which we shall also denote by T. That is

$$T_{ij} = \begin{cases} 1 & \text{if state } x_i \text{ is the successor to state } x_j, \\ & \text{i.e. } Tx_j = x_i \\ 0 & \text{otherwise.} \end{cases}$$

By definition T has exactly one "1" per column. Suppose T results in a cycle space having k transient elements and m cycles of lengths $r_1, r_2, r_3, \ldots, r_m$, respectively. Consider the characteristic polynomial of T, $|T - \lambda I|$ where operations are

performed in the real field. It is straightforward to show that this polynomial will have the form

$$\pm \lambda^k \prod_{i=1}^{m} (\lambda^{r_i} - 1)$$

(see Cull for further details).

In the T matrix we can see that we have $T_{ii} = 1$ i.f.f. state i is on a cycle of length 1. Thus $Tr(T)$ gives the number of elements on cycles of length 1. Extending this notion it is apparent that

(1) $Tr(T^m)$ = number of states on cycles whose length divides m .

Hence $Tr(T^{n!})$ equals the number of states on cycles and $n - Tr(T^{n!})$ equals the number of transient states.

It is of interest to obtain a matrix A_m from T such that

$Tr(A_m)$ = number of states on cycles whose length is exactly m .

Let

C_m = {primes $\leq$ m which appear in the prime representation of m}
(i.e. appear with a power $\geq$ 1)

and let

N_m = number of elements in C_m .

The number of subsets of N_m is 2^{N_m} and the number of subsets of size k is $\binom{N_m}{k} \equiv N_{mk}$. At a given k let j index the subsets of size k so that the 2^{N_m} subsets may be denoted by C_{kj} , $k = 0,1,2,\ldots,N_m$, $j = 1,2,\ldots, N_{mk}$. Let g_{kj} equal m dived by the product of all the elements in C_{kj} . Then

Theorem. For each m , $m = 1,2,\ldots,n$, let

$$A_m = \sum_{k=0}^{N_m} (-1)^k \sum_{j=1}^{N_{mk}} T^{g_{kj}} . \tag{2}$$

Then $Tr(A_m)$ = number of states on cycles whose length is exactly m .

Proof. The most direct proof employs a straightforward, but tedious, inclusion-exclusion argument.

3. Random Transformations.

Consider now the selection of a random (equally likely) transformation T from $\mathcal{J}$. This selection is conveniently accomplished as a sequence of n independent multinomial trials where the j^{th} trial chooses the successor to state j in an equiprobable fashion from amongst the n elements in X . This approach clearly results in an equiprobable selection of the n^n elements in $\mathcal{J}$.

Then $Tr(T)$, the number of states on cycles of length 1, is obviously distributed as binomial $(n, \frac{1}{n})$ with $E(Tr(T)) = 1$, $var(Tr(T)) = (n-1)/n$. The probability that T has no cycles of

length 1 is $((n-1)/n)^n$; the probability that state i is a successor state is $1-((n-1)/n)^n$. As $n \to \infty$ these probabilities tend to e^{-1} and $1 - e^{-1}$, respectively. More generally the limiting distribution of Tr(T) is Poisson (1).

We now examine the nature of the cycle space of a random transformation. In particular, we pose the following questions.

(i) What is the probability that state x_i is on a cycle of length r ?

(ii) What is the joint probability that state x_i is on a cycle of length r and state x_j is on a cycle of length s ?

(iii) What is the expected number of cycles of length r and the expected number of states on cycles of length r ?

(iv) What is the distribution of the number of cycles of length r and of the number of states on cycles of length r ?

(v) What is the joint distribution of the number of cycles of length r and the number of cycles of length s ? of the number of states on cycles of length r and the number of states on cycles of length s ?

(vi) What is the expected number of cycles and the expected number of states on cycles?

(vii) What is the distribution of the number of cycles and of the number of states on cycles?

(viii) What is the expected length of a cycle?

In what follows we shall provide exact or asymptotic answers to all of these questions. Some aspects of this distribution theory (e.g. (iv), (vii) and (viii)) have been studied by Rubin and Sitgreaves and by Cull. However, the distribution of $Tr(T^k)$ and $Tr(A_m)$ as in (1) and (2) are extremely difficult to examine directly. In the next section we will show how an approach using a sequence of square arrays can be employed advantageously in answering the above questions.

4. Square Arrays.

For a set X of n elements and T selected at random from $\mathcal{J}$ consider the $n \times n$ array of random variables.

$$(3) \qquad \begin{matrix} D^n_{11} & \cdots\cdots & D^n_{1n} \\ D^n_{21} & \cdots\cdots & D^n_{2n} \\ \vdots & & \vdots \\ D^n_{n1} & \cdots\cdots & D^n_{nn} \end{matrix}$$

where

$$D^n_{ri} = \begin{cases} 1 & \text{if state } x_i \text{ is on a cycle of length } r \\ 0 & \text{otherwise.} \end{cases}$$

From this array we are interested in the following variables.

$$(4) \qquad S_{n,r} = \sum_{i=1}^{n} D^n_{ri} = \text{number of states on a cycle of length } r$$

(5) $T_{n,r} = S_{n,r}/r$ = number of cycles of length r

(6) $C_i^n = \sum_{r=1}^{n} D_{ri}^n = \begin{cases} 1 & \text{if state } x_i \text{ is on a cycle} \\ 0 & \text{otherwise} \end{cases}$

(7) $U_n = \sum_{r=1}^{n} S_{n,r} = \sum_{i=1}^{n} C_i^n$ = number of states on cycles

(8) $V_n = \sum_{r=1}^{n} T_{n,r}$ = number of cycles.

Note that while a row sum $(S_{n,r})$ may exceed 1, by definition the column sums (C_i^n) are still 0-1 random variables. In fact, $P(C_i^n = 0)$ is the probability that state i is transient.

For any fixed r the joint distribution of $D_{r1}^n, \ldots, D_{rn}^n$ or of any subset will be that of a collection of dependent interchangeable random variables. The marginal distribution of any D_{ri}^n is given by

(9) $$P(D_{ri}^n = 1) = P(\text{state } x_i \text{ is on a cycle of length } r)$$

$$= \binom{n-1}{r-1}(r-1)!\ \frac{1}{n^r} = \frac{1}{n}\ \frac{(n)_r}{n^r}$$

where $(n)_r$ is the falling factorial of r terms starting at n. Thus we immediately have $E(D_{ri}^n)$ and $\mathrm{var}(D_{ri}^n)$ and may note that as $n \to \infty$ both tend to 0.

We can immediately obtain the expectation of the variables in (4) through (8), i.e.

(10) $$E(S_{n,r}) = (n)_r/n^r$$

$$(11) \qquad E(T_{n,r}) = \frac{1}{r} (n)_r / n^r$$

$$(12) \qquad E(C_1^n) = \frac{1}{n} \sum_{r=1}^{n} (n)_r / n^r$$

$$(13) \qquad E(U_n) = \sum_{k=1}^{n} (n)_r / n^r$$

$$(14) \qquad E(V_n) = \sum_{r=1}^{n} \frac{1}{r} (n)_r / n^r$$

The limits of (10) and (11) are clearly 1 and $1/r$, respectively. By truncating the sums at arbitrary m and letting $n \to \infty$, the limits in (13) and (14) are both seen to be ∞. For (12) the limit is 0, i.e. fixing $m < n$ we have

$$\frac{1}{n} \sum_{r=1}^{n} \frac{(n)_r}{n^r} \leq \frac{1}{n} \sum_{i=1}^{m-1} \frac{(n)_r}{n^r} + \frac{(n)_m}{n^{m+1}} \sum_{r=m}^{n} \left(\frac{n-m}{n}\right)^{r-m}$$

$$\leq \frac{1}{n} \sum_{i=1}^{m-1} \frac{(n)_r}{n^r} + \frac{(n)_m}{n^{m+1}} \frac{1 - \left(\frac{n-m}{n}\right)^{n-m+1}}{1 - \left(\frac{n-m}{n}\right)}$$

$$\leq \frac{1}{n} \sum_{i=1}^{m-1} \frac{(n)_r}{n^r} + \frac{(n)_m}{mn^m} \left[1 - \left(1 - \frac{m}{n}\right)^{n-m+1}\right] .$$

As $n \to \infty$ the right-hand side approaches $\frac{1}{m}(1 - e^{-m})$. But m is arbitrary so that the limit of the left-hand side must be 0.

The interpretation of these limits is that (i) the probability of any particular state being on a cycle tends to 0 with increasing number of states, but (ii) the expected number of cyclic states and expected number of cycles tends to ∞ with increasing number of states.

Consider the joint distribution of any pair, D^n_{ri} , D^n_{sj} . We have three cases: (i) $r \neq s$, $i \neq j$, (ii) $r = s$, $i \neq j$, (iii) $r \neq s$, $i = j$.

For (i) we have

$$(15)\quad P(D^n_{ri} = 1 \ , D^n_{sj} = 1) = \begin{cases} \dfrac{1}{n(n-1)} \dfrac{(n)_{r+s}}{n^{r+s}} , & r + s \leq n \\ 0 , & r + s > n \end{cases}$$

For (ii) we have

$$(16)\quad P(D^n_{ri} = 1 \ , D^n_{rj} = 1) = \begin{cases} \dfrac{(r-1)}{n(n-1)} \dfrac{(n)_r}{n^r} + \dfrac{1}{n(n-1)} \dfrac{(n)_{2r}}{n^{2r}} , & 2r \leq n \\ \dfrac{(r-1)}{n(n-1)} \dfrac{(n)_r}{n^r} , & 2r > n \end{cases}$$

For (iii) we have two exclusive events so that

$$(17)\quad P(D^n_{ri} = 1 \ , D^n_{si} = 1) = 0 .$$

In each case using (9) we may obtain expressions for the three remaining joint events.

Continuing we have in case (i)

$$(18)\quad \operatorname{cov}(D^n_{ri} \ , D^n_{sj}) = \begin{cases} \dfrac{1}{n(n-1)} \dfrac{(n)_{r+s}}{n^{r+s}} - \dfrac{1}{n^2} \dfrac{(n)_r(n)_s}{n^{r+s}} , & r + s \leq n \\ - \dfrac{1}{n^2} \dfrac{(n)_r(n)_s}{n^{r+s}} , & r + s > n , \end{cases}$$

in case (ii)

$$(19)\quad \operatorname{cov}(D^n_{ri},D^n_{rj}) = \begin{cases} \frac{r-1}{n(n-1)} \frac{(n)_r}{n^r} + \frac{1}{n(n-1)} \frac{(n)_{2r}}{2r} - \frac{1}{n^2} \frac{[(n)_r]^2}{n^{2r}} \;, & 2r \le n \\ \frac{(r-1)}{n(n-1)} \frac{(n)_r}{n^r} - \frac{1}{n^2} \frac{[(n)_r]^2}{n^{2r}} \;, & 2r > n \end{cases}$$

and in case (iii)

$$(20)\quad \operatorname{cov}(D^n_{ri},D^n_{si}) = -\frac{1}{n^2} \frac{(n)_r(n)_s}{n^{r+s}} \;.$$

In all cases these covariances tend to 0 as $n \to \infty$, a fact which could be inferred without computation from the Cauchy-Schwarz inequality and (9).

Hence

$$(21)\quad \operatorname{cov}(S_{n,r},S_{n,s}) = \begin{cases} \frac{(n)_{r+s}}{n^{r+s}} - \frac{(n)_r(n)_s}{n^{r+s}} \;, & r \neq s \;,\; r+s \le n \\ -\frac{(n)_r(n)_s}{n^{r+s}} \;, & r \neq s \;,\; r+s > n \end{cases}$$

$$(22)\quad \operatorname{cov}(T_{n,r},T_{n,s}) = \frac{1}{rs} \operatorname{cov}(S_{n,r},S_{n,s})$$

$$(23)\quad \operatorname{var}(S_{n,r}) = \begin{cases} r\frac{(n)_n}{n^r} + \frac{(n)_{2r}}{n^{2r}} - \frac{[(n)_r]^2}{n^{2r}} \;, & 2r \le n \\ r\frac{(n)_r}{n^r} - \frac{[(n)_r]^2}{n^{2r}} \;, & 2r > n \end{cases}$$

$$(24)\quad \operatorname{var}(T_{n,r}) = \frac{1}{r^2} \operatorname{var}(S_{n,r})$$

$$(25)\quad \operatorname{cov}(C_i^n, C_j^n) = \frac{2}{n(n-1)} \sum_{r=1}^{n} (r-1) \frac{(n)_r}{n^r} - \frac{1}{n^2} \left(\sum_{r=1}^{n} \frac{(n)_r}{n^r} \right)^2$$

$$(26)\quad \operatorname{var}(C_i^n) = \frac{1}{n} \sum_{r=1}^{n} \frac{(n)_r}{n^r} - \frac{1}{n^2} \left| \sum_{r=1}^{n} \frac{(n)_r}{n^r} \right|^2$$

$$(27)\quad \operatorname{var}(U_n) = \sum_{r=1}^{n} (2r-1) \frac{(n)_r}{n^r} - \left(\sum_{r=1}^{n} \frac{(n)_r}{n^r} \right)^2$$

$$(28)\quad \operatorname{var}(V_n) = \sum_{r=1}^{n} \frac{1}{r} \frac{(n)_r}{n^r} - \left(\sum_{r=1}^{n} \frac{1}{r} \frac{(n)_r}{n^r} \right)^2 + \sum_{\substack{r,s \geq 1 \\ r+s \leq n}} \frac{1}{rs} \frac{(n)_{r+s}}{n^{r+s}} .$$

From these expressions it is clear that $S_{n,r}$ and $S_{n,s}$ (also $T_{n,r}$ and $T_{n,s}$) are always negatively correlated but asymptotically uncorrelated. Also $\lim_{n \to \infty} \operatorname{var}(S_{n,r}) = r$, $\lim_{n \to \infty} \operatorname{var}(T_{n,r}) = 1/r$. It is also apparent that $\lim_{n \to \infty} \operatorname{var}(C_i^n) = 0$ and thus that $\lim_{n \to \infty} \operatorname{cov}(C_i^n, C_j^n) = 0$. Finally, $\operatorname{var}(U_n)$ and $\operatorname{var}(V_n)$ both tend to ∞ as $n \to \infty$, as will be most easily seen from results in section 6.

Extending cases (i), (ii) and (iii) above, consider any subset of size m of the D_{ri}^n . Suppose first that all m variables are in the same row of (3). Taking $mr \leq n$ and recognizing the exchangeability of the variables, we seek

$$\begin{aligned} P_{n,m,r} &= P(D_{r\alpha_1}^n = D_{r\alpha_2}^n = \dots = D_{r\alpha_m}^n = 1) \\ &= P(\text{states } x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_m} \text{ are each on a cycle of length } r). \end{aligned}$$

To obtain an expression for this probability, consider all possible partitions of m with no part greater than r. If a given partition has parts $m_1,\dots,m_j$, let $n(m_1,\dots,m_j)$ be the number of ways to allocate m distinct objects into j like cells with m_i in cell i $(\Sigma_{i=1}^{j} m_i = m)$. Also associate with $m_1,m_2,\dots,m_j$ the event $A_{nr}(m_1,\dots,m_j)$ defined by {states $x_{\alpha_1},\dots,x_{\alpha_{m_1}}$ on the same cycle of length r, states $x_{\alpha_{m_1+1}},\dots,$ $x_{\alpha_{m_1+m_2}}$ on the same cycle of length r, etc.}. If $\mathcal{S}_m$ is the set of all partitions of m and $\mathcal{S}_{m,r}$ is the set of all partitions of m with no part greater than r, then

$$(29) \qquad P_{n,m,r} = \sum_{\mathcal{S}_{m,r}} n(m_1,\dots,m_j)\, P(A_{nr(m_1,m_2,\dots,m_j)})$$

with

$$(30) \qquad P(A_{nr(m_1,m_2,\dots,m_j)}) = \frac{1}{(n)_m} (n)_{jr}\; n^{-jr}\, [(r-1)!]^j \, [\prod_{i=1}^{j} (r-m_i)!]^{-1}$$

Using (29) with appropriate subsets of size $m-1$, we may in principle obtain the complete joint distribution of the m $D^n_{r\alpha_i}$.

If on the other hand the m D^n_{ri} are all in the same column of (3), say $D^n_{\alpha_1 i},\dots,D^n_{\alpha_m i}$, in accordance with (17) their joint distribution will be multinomial with associated $P_{\alpha_j} = \frac{1}{n} \frac{(n)_{\alpha_j}}{n^{\alpha_j}}$, $j = 1,\dots,m$.

Extending the above ideas, we may obtain the joint distribution of any subset of size m of D^n_{ri}.

5. Exact Distributions

Returning to the variables in (4) - (8), we have already noted that C_1^n is a 0-1 variable with success probability given by (12).

Next we obtain the exact distributions of U_n following ideas given by Rubin and Sitgreaves. Given T, for any $x \in X$, we can define the set of all successors to x, $S(x)$, i.e.

$$S(x) = \{x' : T^r x = x' \text{ for some } r \geq 0\}.$$

By definition $x \in S(x)$ and $S(x)$ includes all the cyclic states associated with x (although x is, of course, not necessary cyclic). Then with $k \geq r + 1$

$$\begin{aligned}
&P(x \text{ has } k \text{ successors, } S(x) \text{ has cycle of length } r,\ x \text{ is not cyclic}) \\
&= P(Tx \neq x\ ;\ T^2x \neq Tx\ ,\ T^2x \neq x\ ;\ T^3x \neq T^2x\ ,\ T^3x \neq Tx\ ,\ T^3x \neq x\ ; \\
&\qquad T^{k-1}x \neq T^{k-2}x, \ldots, T^{k-1}x \neq x\ ;\ T^kx = T^{k-r}x) \\
&= \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-(k-1)}{n} \cdot \frac{1}{n} \\
&= \frac{(n)_k}{n^{k+1}} .
\end{aligned}$$

Thus

$$P(S(x) \text{ has cycle of length } r,\ x \text{ is not cyclic}) = \sum_{k=r+1}^{n} \frac{(n)_k}{n^{k+1}} . \tag{31}$$

But

P(S(x) has cycle of length r, x is not cyclic)

$$= \sum_{u=r}^{n} P(S(x) \text{ has cycle of length } r, x \text{ is not cyclic}, U_n = u)$$

$$= \sum_{u=r}^{n} P(S(x) \text{ has cycle of length } r \mid x \text{ is not cyclic}, U_n = u)$$

$$P(x \text{ is not cyclic} \mid U_n = u)\ P(U_n = u)$$

$$= \sum_{u=r}^{n} \left(1 \cdot \frac{u-1}{u} \cdot \frac{u-2}{u-1} \cdots \frac{u-(r-1)}{u-(r-2)} \cdot \frac{1}{u-(r-1)}\right) \cdot \frac{n-u}{n} \cdot P(U_n = u)$$

$$(32) \qquad = \sum_{u=r}^{n} \frac{n-u}{nu} P(U_n = u) .$$

Now (31) and (32) are equal for all r implying

$$\sum_{k=r+1}^{n} \frac{(n)_k}{n^{k+1}} - \sum_{k=r+2}^{n} \frac{(n)_k}{n^{k+1}} = \sum_{u=r}^{n} \frac{n-u}{nu} P(U_n = u) - \sum_{u=r+1}^{n} \frac{n-u}{nu} P(U_n = u)$$

from which

$$(33) \qquad P(U_n = u) = \frac{(n)_u u}{n^{u+1}} , \ u = 1,2,\ldots,n .$$

From (33), $P(U_n = n) = \frac{n!}{n^n}$. This is seen directly by noting that $U_n = n$ i.f.f. T is 1-1 and that there are $n!$ such T . Harris offers an alternative development of (33) by decomposing the cycle space of T and employing a convenient identity from Katz.

Using (33) we have the identity

$$(34) \qquad \sum_{u=1}^{n} \frac{(n)_u}{n^u} u = n$$

Taking the mean of U_n from (33) and equating to (13) we have the identity

$$\sum_{u=1}^{n} \frac{(n)_u}{n^u} = \frac{1}{n} \sum_{u=1}^{n} \frac{(n)_u}{n^u} u^2 \quad \text{or} \quad n\, E(\frac{1}{U_n}) = E(U_n) \; . \tag{35}$$

Continuing in this fashion, from (27) we have

$$E(U_n^2) = \sum_{u=1}^{n} (2u-1) \frac{(n)_u}{n^u} = 2n - \sum_{u=1}^{n} \frac{(n)_u}{n^u} \quad \text{or} \tag{36}$$

$$E(U_n^2) = 2n - E(U_n)$$

and hence the identity

$$\sum_{u=1}^{n} \frac{(n)_u u^3}{n^u} = 2n^2 - n \sum_{u=1}^{n} \frac{(n)_u}{n^u} = 2n^2 - \sum_{u=1}^{n} \frac{(n)_u}{n^u} u^2 \; . \tag{37}$$

Note that $n^{-1}E(U_n^2) \to 2$.

The exact distribution of V_n is obtained from U_n by

$$P(V_n = v) = \sum_{u=v}^{n} P(V_n = v | U_n = u)\, P(U_n = u)$$

$$= \sum_{u=v}^{n} \alpha_n(u,v) \frac{(n)_u u}{n^{u+1}} \; . \tag{38}$$

But it is clear that α does not depend upon n . It is just the probability of exactly v cycles resulting from u cyclic elements. In fact, we may show (Riordan p. 70-72) that

$$\alpha(u,v) = (-1)^{u+v}\, s(u,v)/u!$$

where $s(u,v)$ are Stirling numbers of the first kind.

$$\alpha(1,1) = 1$$

$$\alpha(2,1) = 1/2 \qquad \alpha(2,2) = 1/2$$

$$\alpha(3,1) = 1/3 \qquad \alpha(3,2) = 1/2 \qquad \alpha(3,3) = 1/6 \ .$$

More generally $\alpha(u,1) = \frac{1}{u}$, $\alpha(u,u) = \frac{1}{u!}$ and using the familiar recurrence relationship for Stirling numbers of the first kind (Riordan p. 33)

$$(39) \qquad \alpha(u,v) = \frac{u-1}{u}\,\alpha(u-1,v) + \frac{1}{u}\,\alpha(u-1,v-1) \ .$$

Rubin and Sitgreaves tabulate $\alpha(u,v)$ for $u,v = 1,2,\ldots,25$, $u \le v$. The distribution of V_n is obtained in a more complicated form than (38) by Folkert using the aforementioned Katz identity.

Using (14) and (38) the identity (40) ensues

$$(40) \qquad \sum_{u=1}^{n} \frac{1}{u} \frac{(n)_u}{n^u} = \sum_{v=1}^{n} \sum_{u=v}^{n} \alpha(u,v) \frac{(n)_u uv}{n^{u+1}}$$

$$= \sum_{u=1}^{n} \frac{(n)_u}{n^u} \frac{u}{n} \sum_{v=1}^{u} v\,\alpha(u,v)$$

Next the exact distribution of $T_{n,r}$ (equivalently $S_{n,r}$ since $P(S_{n,r} = kr) = P(T_{n,r} = k)$) is obtained from U_n .

$$P(T_{n,r} = k) = \sum_{u=kr}^{n} P(T_{n,r} = k \mid U_n = u)\, P(U_n = u)$$

$$(41) \qquad = \sum_{u=kr}^{n} \beta_n(r,k,u) \cdot \frac{(n)_u u}{n^{u+1}} \ .$$

Now β does not depend on n . It is just the probability of exactly k cycles of length r resulting from u cyclic elements. It is not hard to show that

$$\beta(r,k,u) = \frac{1}{k!r^k}\,\beta(r,0,u-kr) \; . \tag{41}$$

Since $\beta(r,0,w) = 1 - \sum_{k=1}^{[w/r]} \beta(r,k,w)$ and since $\beta(r,0,w) = 1$ when $w < r$, $\beta(r,k,u)$ can be obtained recursively. Also $\beta(r,1,r) = \alpha(1,r) = 1/r$ and $\beta(1,r,r) = \alpha(r,r) = 1/r!$

It is apparent that with the exception of U_n , these exact distributions are a bit inconvenient. In the next section we obtain some simple asymptotic distributions.

In concluding this section we examine the expected length of a cycle denoted by ECL. We first compute the likelihood of any particular cycle space configuration under a random T . If we let m_ℓ be the number of cycles of length ℓ , $\ell = 1,\ldots,n$, and let $m_0 = n - \Sigma m_\ell \ell$ = the number of transient states, then for $\Sigma m_\ell \ell \le n$

$$\begin{aligned}
&P(m_\ell \text{ cycles of length } \ell \text{ and } m_0 \text{ transient states}) \\
&\equiv P(m_1,m_2,\ldots,m_n) \\
&= P(m_1,\ldots,m_2 \mid U_n = n-m_0)\; P(U_n = n-m_0) \\
&= \frac{1}{\prod_{\ell=1}^{n-m_0} m_\ell! \; \prod_{\ell=1}^{n-m_0} \ell^{m_\ell}} \cdot \frac{(n)_{n-m_0}(n-m_0)}{n^{n-m_0-1}}
\end{aligned}$$

$$= \frac{n!\,(n-m_0)}{\prod_{\ell=1}^{n-m_0} \ell^{m_\ell} \prod_{\ell=1}^{n-m_0} m_\ell!\; n^{n-m_0+1}}$$

Given any vector $(m_1,\ldots,m_n)$ such that $m_\ell \geq 0$ and $\Sigma m_\ell \ell \leq n$ the average cycle length for the cycle space configuration it defines is $(\Sigma m_\ell)^{-1}\, \Sigma m_\ell \ell$.

Hence

(43) $$ECL = \sum_{\{(m_1,m_2,\ldots,m_n)=\Sigma m_\ell \leq \ell,\, m_\ell \geq 0\}} (\Sigma m_\ell)^{-1}\, \Sigma m_\ell \ell \cdot P(m_1,m_2,\ldots,m_n)$$

Continuing we note that $\Sigma m_\ell \ell$ is a value of U_n and Σm_ℓ is a value of V_n and thus

$$ECL = E\left(\frac{U_n}{V_n}\right) .$$

Using the joint distribution of U_n, V_n contained in (38), we have

(44) $$ECL = \sum_{v=1}^{n} \sum_{u=v}^{n} \frac{u}{v}\, \alpha(u,v)\, \frac{(n)_u u}{n^{u+1}} = \sum_{u=1}^{n} \sum_{v=1}^{u} \frac{u}{v}\, \alpha(u,v)\, \frac{(n)_u u}{n^{u+1}} .$$

The equality of the right hand sides of (43) and (44) provides yet another identity. A more convenient expression for studying FCL may be obtained using the recursion relation (39). That is,

$$E(V_n) = \sum_{u=1}^{n} \sum_{v=1}^{u} v\,\alpha(u,v)\,\frac{(n)_u u}{n^{u+1}}$$

$$= \sum_{u=1}^{n} \sum_{v=1}^{u} v[(u-1)\alpha(u-1,v) + \alpha(u-1,v-1)] \cdot \frac{(n)_u}{n^{u+1}}$$

$$= \sum_{u=1}^{n} \frac{(n)_u}{n^{u+1}} \cdot \{(u-1)E(v|u-1) + E(v+1|u-1)\}$$

$$= \sum_{u=1}^{n} \frac{(n)_u}{n^{u+1}} + \sum_{u=1}^{n} \frac{(n)_u}{n^{u+1}}\, u\, E(v|u-1)$$

$$= E\left(\frac{1}{U_n}\right) + \frac{1}{n} E\left[(n-U_n) \cdot \left(V_n + \frac{V_n}{U_n}\right)\right] .$$

After some simplification we have

$$E\left(\frac{V_n}{U_n}\right) = E\left[\frac{(U_n+1)V_n}{n}\right] - E\left(\frac{1}{U_n}\right) .$$

Using (35) we obtain

$$E\left(\frac{V_n}{U_n}\right) = \frac{1}{n} E(U_n V_n + V_n - U_n) \tag{45}$$

whence

$$ECL \geq n[E(U_n V_n + V_n - U_n)]^{-1} . \tag{46}$$

6. Asymptotic Results

Using Harris' idea (p. 1047) we obtain the asymptotic probability density of U_n. Letting $W_n = U_n / \sqrt{n}$ and using (33) we may show after some manipulation that W_n converges in distribution to a random variable W having a Rayleigh distribution, i.e. the density of W is

$$f_W(w) = we^{-w^2/2}, \quad w > 0 . \tag{47}$$

This also establishes that $U_n \xrightarrow{P} \infty$.

It is easy to show that

$$E(W^r) = 2^{r/2}\,\Gamma(\tfrac{r+2}{2}) , \quad r > -2 .$$

Thus for $k > -2$

$$E(n^{-k/2} U_n^k) = n^{-k/2} \sum_{u=1}^{n} \frac{u^{k+1}(n)_u}{n^{u+1}} \to 2^{k/2}\,\Gamma(\tfrac{k+2}{2}) .$$

In particular from (35) we have

$$E(\frac{U_n}{\sqrt{n}}) = n^{-1/2} \sum_{u=1}^{n} \frac{(n)_u u^2}{n^{u+1}} = n^{-1/2} \sum_{u=1}^{n} \frac{(n)_u}{n^u} = E(\frac{\sqrt{n}}{U_n}) \to \sqrt{\pi/2} \tag{48}$$

(offering a different verification of our limits for (12) and (13)).

Furthermore, in agreement with our remark after (37) we have

$$n^{-1} \sum_{u=1}^{n} \frac{(n)_u u^3}{n^{u+1}} \to 2 .$$

Expression (48) also implies that the expected number of transient states approaches ∞ as $n \to \infty$, i.e.

$$E(n-U_n) = \sqrt{n}\, E(\sqrt{n} - \frac{U_n}{\sqrt{n}}) \to \infty .$$

Additionally, $\text{var}(\frac{U_n}{\sqrt{n}}) \to 2 - \pi/2$ confirming that $\text{var}(U_n) \to \infty$ as noted after (28). As for $\text{var}(V_n)$, using (28), it is clear that

$$E(V_n^2) = \sum_{r=1}^{n} \frac{1}{r} \frac{(n)_r}{n^r} + \sum_{\substack{r,s \geq 1 \\ r+s \leq n}} \frac{1}{r} \frac{1}{s} \frac{(n)_{r+s}}{n^{r+s}}$$

which is

$$\leq E(V_n) + E(U_n)$$

$$\leq 2E(U_n) \quad \text{since} \quad V_n \leq U_n .$$

Hence from (48) $n^{-1}E(V_n^2) \to 0$ implying $E(n^{-1/2}V_n) \to 0$ and thus that $\text{var}(n^{-1/2}V_n) \to 0$. Similar computation leads to $\text{var}(V_n) \to \infty$.

We now establish that as $n \to \infty$, $ECL \to \infty$. Consider

$$E(U_nV_n) = \sum_{r=1}^{n} \sum_{s=1}^{n} r^{-1} E(S_{n,r}\, S_{n,s}) .$$

From (10), (21) and (23)

$$E(S_{n,r}\ S_{n,s}) = \begin{cases} \dfrac{(n)_{r+s}}{n^{r+s}} , & r \neq s , \quad r + s \leq n \\[2ex] 0 , & r \neq s , \quad r + s > n \\[2ex] r\ \dfrac{(n)_r}{n^r} + \dfrac{(n)_{2r}}{n^{2r}} , & r = s , \quad 2r \leq n \\[2ex] r\ \dfrac{(n)_r}{n^r} , & r = s , \quad 2r > n \end{cases}$$

Hence

$$E(U_n\ V_n) = \sum_{\substack{r\geq 1, s\geq 1 \\ r+s\leq n}} r^{-1} \frac{(n)_{r+s}}{n^{r+s}} + \sum_{r=1}^{n} \frac{(n)_r}{n^r}$$

which may be shown to be

$$(49) \qquad \leq n\ E\left(\frac{1 + \log U_n}{U_n}\right) + E(U_n) .$$

Upon dividing by n both terms on the right-hand side of (49) approach 0 . For the first we use the boundedness of the argument and (47) while for the second we use (48) again. As a result $n^{-1}E(U_n\ V_n) \to 0$ so that from (46) $ECL \to \infty$.

We next argue that the asymptotic distribution of $T_{n,r}$ (i.e. $\frac{S_{n,r}}{r}$) is Poisson with mean $1/r$. The limits in (11) and (24) encourage this possibility. It suffices to show that

$$(50) \qquad \lim_{n \to \infty} [k!r^k P(T_{n,r} = k) - P(T_{n,r} = 0)] = 0 , \quad k = 1,2,\ldots .$$

From (40) and (41) we may write

$$P(T_{n,r} = k) = \frac{1}{k!r^k} \sum_{u=0}^{n-kr} \beta(r,0,u) \frac{(n)_{u+kr}(u+kr)}{n^{u+kr+1}} .$$

Hence the left-hand side of (50) becomes

$$\lim_{n \to \infty} [\sum_{u=1}^{n} \beta(r,0,u) \frac{(n)_u u}{n^{u+1}} - \sum_{u=0}^{n-kr} \beta(r,0,u) \frac{(n)_{u+kr}(u+kr)}{n^{u+kr+1}}]$$

$$= \lim_{n \to \infty} [\sum_{u=1}^{n-kr} \beta(r,0,u)\mu(\frac{(n)_u}{n^{u+1}} - \frac{(n)_{u+kr}}{n^{u+kr+1}}) + \sum_{u=n-kr+1}^{n} \beta(r,0,u) \frac{(n)_u u}{n^{u+1}}$$

$$- kr \sum_{u=0}^{n-kr} \frac{(n)_{u+kr}}{n^{u+kr+1}} - \frac{(n)_{kr}(kr)}{n^{kr+1}}] .$$

It is apparent that the limits of the second, third, and fourth terms within the brackets are 0 . Since $\beta(r,0,u) \leq 1$ and since

$$\lim_{n \to \infty} \sum_{u=a}^{n} \frac{(n)_u u}{n^{u+1}} = \lim_{n \to \infty} \sum_{u=1}^{n-a} \frac{(n)_u u}{n^{u+1}} = 1$$

for any fixed positive integer a , the first term also tends to 0 and we are done.

Summarizing, $T_{n,r}$ converges in distribution to a random variable T_r such that

$$P(T_r = t) = \frac{e^{-1/r}}{r^t t!} , \quad t = 0,1,2,\ldots .$$

(The limiting Poisson distribution when $r = 1$ was noted at the beginning of section 3.)

It is well-known that if $X \sim P_0(\lambda)$, then $E(X)_k = \lambda^k$ (see e.g. Johnson and Kotz (1969), p. 90) from which

$$E(X^k) = \sum_{j=1}^{k} S(k,j)\lambda^j$$

where the $S(k,j)$ are Stirling numbers of the second kind. Hence

(51) $$E(T_{n,r})^k \to \sum_{j=1}^{k} S(k,j)r^{-j} .$$

We calculate the left-hand side of (51) assuming $n > k$

$$E(T_{n,r})^k = r^{-k} E(\sum_{i=1}^{n} D_{ri}^n)^k$$

$$= r^{-k} \sum_{\mathcal{R}} \frac{k!}{\Pi k_i!} E \, \Pi(D_{ri}^n)^{k_i}$$

where $\mathcal{R} = \{(k_1,\ldots,k_n): k_i \geq 0 , \Sigma k_i = k\}$.
But if exactly m of the $k_i \neq 0$, $E \, \Pi(D_{ri}^n)^{k_i} = P_{n,m,r}$ given by (29). Continuing then

$$E(T_{n,r})^k = r^{-k} \sum_{m=1}^{k} P_{n,m,r} \sum_{\mathcal{R}_m} \frac{k!}{\Pi k_i!}$$

where $\mathcal{R}_m$ denotes the subset of $\mathcal{R}$ on which exactly m of the k_i's are > 0. But the sum over $\mathcal{R}_m$ is merely the number of ways of placing k objects into n cells such that exactly m are nonempty. This number is $(n)_m S(k,m)$ (Riordan, p. 92) whence

$$E(T_{n,r})^k = r^{-k} \sum_{m=1}^{k} S(k,m)(n)_m P_{n,m,r} .$$

Using (30) we have

$$(52) \qquad \lim_{n \to \infty} E(T_{n,r})^k = r^{-k} \sum_{m=1}^{k} S(k,m) \sum_{\mathcal{S}_{m,r}} n(m_1,\ldots,m_j) \cdot$$

$$[(r-1)!]^j [\prod_{i=1}^{j} (r-m)!]^{-1} .$$

Denoting the sum over $\mathcal{S}_{m,r}$ by $\Delta_{r,m}$ and equating right-hand sides in (51) and (52), we find the identity

$$(53) \qquad \sum_{m=1}^{k} S(k,m) r^{k-m} = \sum_{m=1}^{k} S(k,m)\Delta_{r,m} .$$

Note that $\Delta_{1,m} = 1$ reduces (53) to a triviality.

7. Summary

In the previous sections we have rather thoroughly described the behavior of the cycle space of a randomly selected transformation on a finite set. Amongst the most interesting conclusions are the "large set" results. We have demonstrated that with increasing set size

(i) the expected number of cyclic states $\to \infty$.

(ii) the expected number of transient states $\to \infty$.

(iii) the expected number of cycles $\to \infty$.

(iv) the likelihood that any particular state is cyclic $\to 0$.

(v) the expected number of cycles of length $r \to 1/r$.

(vi) the expected number of states on cycles of length $r \to 1$.

(vii) the expected cycle length $\to \infty$.

As a final remark, suppose the set of transformations is restricted to be into a subset of X, say X', having n' elements. After the first transition, all of the results of the preceding sections apply with n' replacing n.

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#305

ON THE CYCLIC BEHAVIOR OF RANDOM TRANSFORMATIONS ON A FINITE SET

Let X be a finite set of elements and let $\mathfrak{T}$ be the set of all transformations from X into itself. For $T \in \mathfrak{T}$ we take T^k to have its usual meaning. Starting from a given $x \in X$ consider the sequence $T^j x$, $j = 0,1,2,\ldots$. Since X is finite the sequence T_j must eventually encounter an element it had shown before. Thereafter it must repeat this intermediate sequence of elements. Such a sequence is called a cycle and the number of distinct elements in the cycle is called the cycle length. For a given T, not all x's must be on a cycle and, moreover, starting from differing x's may lead T into differing cycles. Hence we have the notion of a cycle space for T.

It is the purpose of this paper to discuss a collection of exact and asymptotic results describing the cycle space of a randomly selected T. In particular, we examine such variables as the number of elements on a cycle of a specified length, the number of elements on cycles, the number of cycles and the length of a cycle.

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